1. cvičení - řešení

Pozor: pravděpodobně na několik místech chybí + konstanta - pokud takové místo objevíte, dejte mi, prosím, vědět mailem nebo zpětnou vazbou:

Příklad 1 (a) $f(x) = (x+2)^5$

$$\begin{aligned}
\int f(x)\mathrm{d}x &= \int \left((x+2)^2\right)^2 (x+2)\mathrm{d}x = \int \left(x^2+4x+4\right)^2 (x+2)\mathrm{d}x = \\
&= \int \left(x^4+4x^3+4x^2+4x^3+16x^2+16x+4x^2+16x+16\right)(x+2)\mathrm{d}x = \\
&= \int \left(x^4+8x^3+24x^2+32x+16\right)(x+2)\mathrm{d}x = \\
&= \int x^5+8x^4+24x^3+32x^2+16x+2x^4+16x^3+48x^2+64x+32 \ \mathrm{d}x = \\
&= \int x^5+10x^4+40x^3+80x^2+80x+32 \ \mathrm{d}x = \\
&= \frac{x^6}{6}+\frac{10x^5}{5}+40\frac{x^4}{4}+80\frac{x^3}{3}+80\frac{x^2}{2}+32x+c = \\
&= \frac{x^6}{6}+2x^5+10x^4+\frac{80}{3}x^3+40x^2+32x+c
\end{aligned}$$

Alternativně to lze počítat pomocí substituce. Vyjde $\frac{1}{6}(x+2)^6$ a výpočtem lze zkázat, že jde o tentýž výraz.

Příklad 1 (b) $f(x) = \cos 4x + \dfrac{6}{\sqrt{1-x^2}}$

Připomeňme: $\int \frac{1}{\sqrt{1-x^2}} = \arcsin x$ pro $x \in (-1,1)$ a $(\sin x)' = \cos x$

$$\int f(x)\mathrm{d}x \overset{\text{lin.}}{=} \int \cos(4x)\mathrm{d}x + 6\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\sin(4x)}{4} + 6\arcsin x + c$$

Příklad 1 (c) $f(x) = \dfrac{x^2+1}{x}$

$$\int f(x)\mathrm{d}x = \int x + \frac{1}{x}\mathrm{d}x = \frac{x^2}{2} + \log x + c$$

Příklad 2 (a) $f(x) = \frac{e^x}{e^x+1}$

$$\int f(x)\mathrm{d}x = |y = e^x+1,\ \mathrm{d}y = e^x\mathrm{d}x| = \int \frac{1}{y}\mathrm{d}y = \log y + c = \log(e^x+1)+c$$

Příklad 2 (c) $f(x) = xe^{x^2}$

$$\int f(x)\mathrm{d}x = \left|y = e^{x^2},\ \mathrm{d}y = 2xe^{x^2}\right| = \int \frac{2xe^{x^2}}{2}\mathrm{d}x = \int \frac{1}{2}\mathrm{d}y = \frac{y}{2} + c = \frac{e^{x^2}}{2} + c$$

Příklad 2 (d) $f(x) = (e^x)^2$

$$\int f(x)\mathrm{d}x = |y = e^x,\ \mathrm{d}y = e^x\mathrm{d}x| = \int y\mathrm{d}y = \frac{y^2}{2} + c = \frac{e^{2x}}{2} + c$$

Příklad 2 (e) $f(x) = \dfrac{\sin\log x}{x}$

$$\int f(x)\mathrm{d}x = \left|y = \log x,\ \mathrm{d}y = \frac{1}{x}\mathrm{d}x\right| = \int \sin y\mathrm{d}y = -\cos y + c = -\cos(\log x) + c$$

Příklad 2 (h) $f(x) = \dfrac{\cos 2x}{\sqrt{\sin x\cos x}}$

$$\int f(x)\mathrm{d}x = \left|y = \frac{\sin 2x}{2} = \sin x\cos x,\ \mathrm{d}y = \cos(2x)\mathrm{d}x\right| = \int \frac{1}{\sqrt{y}} = 2\sqrt{y} + c = 2\sqrt{\sin x\cos x} + c$$

Příklad 2 (i) $f(x) = \dfrac{e^{2x}}{\sqrt{e^x - 1}}$

$$\int f(x)\mathrm{d}x = |y = e^x,\ \mathrm{d}y = e^x\mathrm{d}x| = \int \frac{y}{\sqrt{y-1}}\mathrm{d}y = |z = y - 1,\ \mathrm{d}z = \mathrm{d}y| = \int \frac{z+1}{\sqrt{z}}\mathrm{d}z =$$
$$= \int \sqrt{z} + \frac{1}{\sqrt{z}}\mathrm{d}z = \frac{2z^{\frac{3}{2}}}{3} + 2\sqrt{z} + c = \frac{2}{3}(y-1)^{\frac{2}{3}} + 2\sqrt{y-1} + c = \frac{2}{3}(e^x - 1)^{\frac{2}{3}} + 2\sqrt{e^x - 1} + c$$

Příklad 2 (k) $f(x) = \dfrac{x+1}{x^2+3}$

$$\int f(x)\mathrm{d}x \overset{\text{lin.}}{=} \int \frac{x}{x^2+3}\mathrm{d}x + \int \frac{1}{x^2+3}\mathrm{d}x = \left|y = x^2 + 3,\ \mathrm{d}y = 2x\mathrm{d}x\right| = \int \frac{1}{2y}\mathrm{d}y + \int \frac{1}{3}\cdot\frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1}\mathrm{d}x =$$
$$= \left|z = \frac{x}{\sqrt{3}},\ \mathrm{d}z = \frac{1}{\sqrt{3}}\mathrm{d}x\right| = \frac{1}{2}\log y + c + \frac{\sqrt{3}}{3}\int \frac{1}{z^2+1}\mathrm{d}z \overset{(☎)}{=} \frac{1}{2}\log y + \frac{\sqrt{3}}{3}\arctan z + c =$$
$$= \frac{1}{2}\log\left(x^2 + 3\right) + \frac{\sqrt{3}}{3}\arctan\frac{x^2+3}{\sqrt{3}} + c$$

(☎) $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + c$

Příklad 2 (n) $f(x) = \dfrac{7}{\cos x}$

$$\int f(x)\mathrm{d}x = \int \frac{7\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{7\cos x}{1-\sin^2 x}\mathrm{d}x = |y=\sin x,\ \mathrm{d}y = \cos x\mathrm{d}x| = \int \frac{7}{1-y^2}\mathrm{d}y =$$
$$\overset{\text{Př. 4a}}{=} \frac{7}{2}\left(\log(|y+1|) - \log(|y-1|)\right) + c = \frac{7}{2}\left(\log(|\sin x+1|) - \log(|\sin x-1|)\right) + c$$

Příklad 2 (o) $f(x) = \dfrac{2}{x\sqrt{x^2+1}}$

$$\int f(x)\mathrm{d}x = \left|y=\sqrt{x^2+1},\ \mathrm{d}y = \frac{x}{\sqrt{x^2+1}}\mathrm{d}x\right| = \int \frac{2x}{x^2\sqrt{x^2+1}}\mathrm{d}x = \int \frac{1}{y^2-1}\mathrm{d}y =$$
$$\overset{\text{Př. 4a}}{=} \frac{1}{2}\log\left|\frac{1-y}{1+y}\right| + c = \frac{1}{2}\log\left|\frac{1-\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right| + c$$

Příklad 2 (p) $f(x) = \dfrac{1}{\sqrt{1+e^{2x}}}$

$$\int f(x)\mathrm{d}x = |y=e^x,\ \mathrm{d}y = e^x\mathrm{d}x| = \int \frac{e^x}{e^x\sqrt{1+e^{2x}}}\mathrm{d}x = \int \frac{1}{y\sqrt{1+y^2}}\mathrm{d}y =$$
$$= \left|z=\sqrt{1+y^2},\ \mathrm{d}z = \frac{y}{\sqrt{y^2+1}}\mathrm{d}y\right| = \int \frac{y}{y^2\sqrt{y^2+1}}\mathrm{d}y = \int \frac{1}{z^2-1}\mathrm{d}z \overset{\text{Př. 4a}}{=} \frac{1}{2}\log\left|\frac{1-z}{1+z}\right| + c =$$
$$= \frac{1}{2}\log\left|\frac{1-\sqrt{1+y^2}}{1+\sqrt{1+y^2}}\right| + c = \frac{1}{2}\log\left|\frac{1-\sqrt{1+e^{2x}}}{1+\sqrt{1+e^{2x}}}\right| + c$$

Příklad 2 (q) $f(x) = \sqrt{1-x^2}$

$$\int f(x)\mathrm{d}x = |x=\sin y,\ \mathrm{d}x = \cos y\mathrm{d}y,\ y=\arcsin x| = \int \sqrt{1-\sin^2 y}\cos y\mathrm{d}y = \int |\cos y|\cos y\mathrm{d}y =$$
$$= \int \cos^2 y\mathrm{d}y \overset{(✈)}{=} \frac{y}{2} + \frac{\cos y\sin y}{2} + c = \frac{1}{2}\left(\arcsin x + \cos(\arcsin x)\,x\right) + c$$

(✈):
$\int \cos^2 x\mathrm{d}x \overset{\text{PP}}{=} \cos x\sin x + c + \int \sin^2 x\mathrm{d}x$
$\int \cos^2 x\mathrm{d}x = \int 1-\sin^2 x\mathrm{d}x = x + c - \int \sin^2 x\mathrm{d}x$
Z předchozích dvou rovností plyne: $2\int \sin^2 x\mathrm{d}x = x - \cos x\sin x + c$, tedy $\int \sin^2 x\mathrm{d}x = \frac{x-\cos x\sin x}{2} + c$.
Pak platí $\int \cos^2 x\mathrm{d}x = \int 1-\sin^2 x\mathrm{d}x = x + c - \int \sin^2 x\mathrm{d}x = c + x - \frac{x-\cos x\sin x}{2} = \frac{x}{2} + \frac{\cos x\sin x}{2} + c$

Příklad 3 (a) $f(x) = \log x$

$$\int f(x)\mathrm{d}x = \int 1 \cdot \log x \mathrm{d}x \overset{\text{PP}}{=} \left|u = \log x,\ v' = 1\right| = x \log x - \int x \frac{1}{x} \mathrm{d}x = x \log x - x + c$$

Příklad 3 (d) $f(x) = \cos^2 x$

$$\int f(x)\mathrm{d}x \overset{\text{PP}}{=} u = \cos x,\ v' = \cos x = \sin x \cos x - \int \sin x(-\sin x)\mathrm{d}x = \sin x \cos x + \int \sin^2 x \mathrm{d}x$$
$$\int f(x)\mathrm{d}x = \int 1 - \sin^2 x \mathrm{d}x = x - \int \sin^2 x \mathrm{d}x$$

Z předchozích dvou rovnic plyne: $\sin x \cos x + \int \sin^2 x \mathrm{d}x = x - \int \sin^2 x \mathrm{d}x$, tedy

$$\int \sin^2 x \mathrm{d}x = \frac{x - \sin x \cos x}{2}.$$

Z toho plyne, že

$$\int f(x)\mathrm{d}x = \int 1 - \sin^2 x \mathrm{d}x = x - \int \sin^2 x \mathrm{d}x = x - \frac{x - \sin x \cos x}{2} = \frac{x + \sin x \cos x}{2} + c$$

Příklad 3 (f) $f(x) = (x^2 + 3x + 3)e^x$

$$\int f(x)\mathrm{d}x \overset{\text{PP}}{=} \left|u = x^2 + 3x + 3,\ v' = e^x\right| = (x^2 + 3x + 3)e^x - \int (2x+3)e^x \mathrm{d}x \overset{\text{PP}}{=} \left|u = 2x + 3,\ v' = e^x\right| =$$
$$= (x^2 + 3x + 3)e^x - (2x+3)e^x + \int 2e^x \mathrm{d}x =$$
$$= (x^2 + 3x + 3 - 2x - 3 + 2)e^x = (x^2 + x + 2)e^x + c$$

Příklad 3 (g) $f(x) = xe^x \cos x$

$$\int f(x)\mathrm{d}x \overset{\text{PP}}{=} \left|u = x, v' = e^x \cos x\right| \overset{(❄)}{=} x \frac{e^x}{2} (\cos x + \sin x) - \int \frac{e^x}{2} (\cos x + \sin x)\, \mathrm{d}x =$$
$$\overset{(❄)}{=} x \frac{e^x}{2} (\cos x + \sin x) - \frac{e^x}{4} (\cos x + \sin x) - \frac{1}{2} \int e^x \sin x \mathrm{d}x =$$
$$\overset{(❀)}{=} x \frac{e^x}{2} (\cos x + \sin x) - \frac{e^x}{4} (\cos x + \sin x) - \frac{1}{2} \frac{e^x}{2} (\sin x - \cos x) + c =$$
$$= \frac{e^x}{2} \left(x (\cos x + \sin x) - \sin x\right) + c$$

(❄):

$$\int e^x \cos x \mathrm{d}x \overset{\text{PP}}{=} \left|u = \cos x,\ v' = e^x\right| = e^x \cos x - \int e^x(-\sin x)\mathrm{d}x =$$
$$\overset{\text{PP}}{=} \left|u = \sin x,\ v' = e^x\right| = e^x \cos x + e^x \sin x - \int e^x \cos x \mathrm{d}x$$

$$\Longrightarrow \int e^x \cos x \mathrm{d}x = \frac{1}{2} e^x (\cos x + \sin x) + c$$

(❀):

$$\int e^x \sin x \mathrm{d}x \overset{\text{PP}}{=} \left| u = e^x,\ v' = \sin x \right| = e^x \sin x - \int e^x \cos x \mathrm{d}x \overset{(❀)}{=} e^x \sin x - \frac{e^x \sin x}{2} - \frac{e^x \cos x}{2} + c =$$
$$= \frac{e^x (\sin x - \cos x)}{2} + c$$

Příklad 3 (h) $f(x) = \cos \log x$

$$\int f(x) \mathrm{d}x \overset{\text{PP}}{=} \left| u = \cos \log x, v' = 1 \right| = x \cos \log x - \int x (-\sin \log x) \frac{1}{x} \mathrm{d}x = x \cos \log x + \int \sin \log x \mathrm{d}x =$$
$$\overset{\text{PP}}{=} \left| u = \sin \log x,\ v' = 1 \right| = x \cos \log x + x \sin \log x - \int x \frac{1}{x} \cos \log x \mathrm{d}x =$$
$$= x (\cos \log x + \sin \log x) - \int f(x) \mathrm{d}x$$

Dostáváme: $\int f(x) \mathrm{d}x = x (\cos \log x + \sin \log x) - \int f(x) \mathrm{d}x$, tedy

$$\int f(x) \mathrm{d}x = \frac{x}{2} (\cos \log x + \sin \log x) + c.$$

Příklad 4 (a) $f(x) = \dfrac{1}{x^2 - 1}$

Parciální zlomky: $x^2 - 1 = (x+1)(x-1)$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx - B$$

Soustava:

$$0 = A + B$$
$$1 = A - B$$

Řešení soustavy:

$$A = \frac{1}{2}$$
$$B = \frac{-1}{2}$$

Integrace:

$$\int f(x)\mathrm{d}x = \frac{1}{2}\int \frac{1}{x-1}\mathrm{d}x - \frac{1}{2}\int \frac{1}{x+1}\mathrm{d}x = |y = x-1,\ \mathrm{d}y = \mathrm{d}x,\ z = x+1,\ \mathrm{d}z = \mathrm{d}x| =$$
$$= \frac{1}{2}\int \frac{1}{y}\mathrm{d}y - \frac{1}{2}\int \frac{1}{z}\mathrm{d}z = \frac{1}{2}\log|y| - \frac{1}{2}\log|z| + c = \frac{1}{2}\log\left|\frac{y}{z}\right| + c = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$

Příklad 4 (b) $f(x) = \dfrac{x}{x^3-1}$

Parciální zlomky: $x^3 - 1 = (x-1)(x^2+x+1)$ a z diskriminantu plyne, že (x^2+x+1) nejde již rozložit na součin.

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Soustava:

$$\begin{aligned} 0 &= A + B \\ 1 &= A - B + C \\ 0 &= A - C \end{aligned}$$

Řešení soustavy:

$$\begin{aligned} A &= \frac{1}{3} \\ B &= \frac{-1}{3} \\ C &= \frac{1}{3} \end{aligned}$$

Integrace:

$$\begin{aligned} \int f(x)\mathrm{d}x &= \frac{1}{3}\int \frac{1}{x-1}\mathrm{d}x + \frac{-1}{3}\int \frac{x-1}{x^2+x+1}\mathrm{d}x = \\ &= \left|y = x-1,\ \mathrm{d}y = \mathrm{d}x,\ z = x^2+x+1,\ \mathrm{d}z = (2x+1)\mathrm{d}x\right| = \\ &= \frac{1}{3}\int \frac{1}{y}\mathrm{d}y - \frac{1}{6}\int \frac{2x+1-1-2}{x^2+x+1}\mathrm{d}x = \frac{1}{3}\log|y| - \frac{1}{6}\int \frac{1}{z}\mathrm{d}z - \frac{1}{6}\int \frac{-3}{x^2+x+1}\mathrm{d}x = \\ &\overset{(\boxtimes)}{=} \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| + \frac{1}{2}\sqrt{\frac{4}{3}}\arctan\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} + c \end{aligned}$$

(✉):

$$\int \frac{1}{x^2+x+1}\mathrm{d}x = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x = \frac{4}{3}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}x = \left| y = \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}},\ \mathrm{d}y = \sqrt{\frac{4}{3}}\mathrm{d}x \right| =$$

$$= \frac{4}{3}\sqrt{\frac{3}{4}}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\sqrt{\frac{4}{3}}\mathrm{d}x = \sqrt{\frac{4}{3}}\int \frac{1}{y^2+1}\mathrm{d}y = \sqrt{\frac{4}{3}}\arctan y + c =$$

$$= \sqrt{\frac{4}{3}}\arctan \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} + c$$

Příklad 4 (d) $f(x) = \dfrac{1}{x(1+x)(1+x+x^2)}$

Parciální zlomky: Opět lze snadno ukázat, že x^2+x+1 nelze rozložit na součin.

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

Soustava:

$$\begin{aligned} 0 &= A+B+C \\ 0 &= 2A+B+C+D \\ 0 &= 2A+B+D \\ 1 &= A \end{aligned}$$

Řešení soustavy:

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \\ D &= -1 \end{aligned}$$

Integrace:
Níže používáme (✉), která je výše.

$$\int f(x)\mathrm{d}x = \int \frac{1}{x}\mathrm{d}x - \int \frac{1}{x+1}\mathrm{d}x - \frac{1}{x^2+x+1}\mathrm{d}x \overset{(✉)}{=} \log|x| - \log|x+1| - \sqrt{\frac{4}{3}}\arctan\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} + c$$

Příklad 4 (f) $f(x) = \dfrac{1}{x^4-1}$

Parciální zlomky: $x^4-1=\left(x^2\right)^2-1=\left(x^2+1\right)\left(x^2-1\right)=\left(x^2+1\right)(x+1)(x-1)$

$$\frac{1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

Soustava:

$$\begin{aligned}0&=A+B+C\\0&=A-B+D\\0&=A+B-C\\1&=A-B-D\end{aligned}$$

Řešení soustavy:

$$\begin{aligned}A&=\frac{1}{4}\\B&=-\frac{1}{4}\\C&=0\\D&=-\frac{1}{2}\end{aligned}$$

Integrace:

$$\int f(x)\mathrm{d}x=\frac{1}{4}\int\frac{1}{x-1}\mathrm{d}x-\frac{1}{4}\int\frac{1}{x+1}\mathrm{d}x-\frac{1}{2}\int\frac{1}{x^2+1}\mathrm{d}x=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+c$$

Příklad 4 (h) $f(x)=\dfrac{1}{x^4+1}$

Parciální zlomky: Dle teorie reciprokých rovnic (jde o 1. druh sudého stupně) vytkneme x^2, provedeme substituci $y=x+\frac{1}{x}$ a upravíme, abychom dostali rozklad na součin.

$$\begin{aligned}x^4+1=x^2\left(x^2+\frac{1}{x^2}\right)&=\left|y=x+\frac{1}{x},\ y^2=x^2+\frac{1}{x^2}+2\right|=x^2\left(y^2-2\right)=x^2\left(y+\sqrt{2}\right)\left(y-\sqrt{2}\right)=\\&=x\left(x+\sqrt{2}+\frac{1}{x}\right)x\left(x-\sqrt{2}+\frac{1}{x}\right)=\left(x^2+x\sqrt{2}+1\right)\left(x^2-x\sqrt{2}+1\right)\end{aligned}$$

Z diskriminantu plyne, že získaný součin nelze dále rozložit.

$$\frac{1}{x^4+1}=\frac{Ax+B}{x^2+x\sqrt{2}+1}+\frac{Cx+D}{x^2-x\sqrt{2}+1}$$

Soustava:

$$
\begin{aligned}
0 &= A + C \\
0 &= -A\sqrt{2} + B + C\sqrt{2} + D \\
0 &= A - B\sqrt{2} + C + D\sqrt{2} \\
1 &= B + D
\end{aligned}
$$

Řešení soustavy:

$$
\begin{aligned}
A &= \frac{1}{2\sqrt{2}} \\
B &= \frac{1}{2} \\
C &= \frac{-1}{2\sqrt{2}} \\
D &= \frac{1}{2}
\end{aligned}
$$

Integrace:

$$
\begin{aligned}
\int f(x)\mathrm{d}x &= \frac{1}{2\sqrt{2}} \int \frac{x+\sqrt{2}}{x^2+x\sqrt{2}+1}\mathrm{d}x - \frac{1}{2\sqrt{2}} \int \frac{x-\sqrt{2}}{x^2-x\sqrt{2}+1}\mathrm{d}x = \\
&= \left| y = x^2 + x\sqrt{2} + 1,\ \mathrm{d}y = 2x + \sqrt{2},\ z = x^2 - x\sqrt{2} + 1,\ \mathrm{d}z = 2x - \sqrt{2}\mathrm{d}x \right| = \\
&= \frac{1}{4\sqrt{2}} \int \frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1} + \frac{\sqrt{2}}{x^2+x\sqrt{2}+1}\mathrm{d}x - \frac{1}{4\sqrt{2}} \int \frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1} - \frac{\sqrt{2}}{x^2-x\sqrt{2}+1}\mathrm{d}x = \\
&= \frac{1}{4\sqrt{2}} \int \frac{1}{y}\mathrm{d}y + \frac{1}{4} \int \frac{1}{x^2+x\sqrt{2}+1}\mathrm{d}x - \frac{1}{4\sqrt{2}} \int \frac{1}{z}\mathrm{d}z - \frac{1}{4} \int \frac{1}{x^2-x\sqrt{2}+1}\mathrm{d}x = \\
&\overset{(✪)}{=} \frac{1}{4\sqrt{2}} \left(\log|y| - \log|z|\right) + \frac{\sqrt{2}\arctan\left(x\sqrt{2}+1\right) - \sqrt{2}\arctan\left(x\sqrt{2}-1\right)}{4} + c = \\
&= \frac{1}{4\sqrt{2}} \log\left|\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right| + \frac{\sqrt{2}}{4}\left(\sqrt{2}\arctan\left(x\sqrt{2}+1\right) - \sqrt{2}\arctan\left(x\sqrt{2}-1\right)\right) + c
\end{aligned}
$$

(✪)

$$
\begin{aligned}
\int \frac{1}{x^2+x\sqrt{2}+1}\mathrm{d}x &= \int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}\mathrm{d}x = 2\int \frac{1}{\left(\frac{x+\frac{\sqrt{2}}{2}}{\sqrt{\frac{1}{2}}}\right)^2 + 1}\mathrm{d}x = 2\int \frac{1}{\left(x\sqrt{2}+1\right)^2+1}\mathrm{d}x = \\
&= \left| y = x\sqrt{2}+1,\ \mathrm{d}y = \sqrt{2}\mathrm{d}x \right| = \sqrt{2}\int \frac{1}{y^2+1}\mathrm{d}y = \\
&= \sqrt{2}\arctan y + c = \sqrt{2}\arctan\left(x\sqrt{2}+1\right) + c
\end{aligned}
$$

$$
\int \frac{1}{x^2-x\sqrt{2}+1}\mathrm{d}x = -\sqrt{2}\arctan\left(x\sqrt{2}-1\right) + c \text{ analogicky}
$$

K Příkladu 5: ***Výpočet parciálních zlomků vynechán, volba goniometrické substituce se řídí postupem z teorie (rozbor racionální fce dvou proměnných R). Dále nezapomínejme, že při substituci $y = \operatorname{tg} x$ nebo $y = \operatorname{tg} \frac{x}{2}$ je potřeba omezit, odkud bereme x.***

Příklad 5 (a) $f(x) = \dfrac{\sin x \cos x}{\sin^2 x + 2\cos^2 x}$

$$\int f(x)\mathrm{d}x = \int \frac{\sin x \cos x}{1+\cos^2 x}\mathrm{d}x = \left| R(u,v) = \frac{uv}{1+v^2},\ y = \cos x,\ \mathrm{d}y = -\sin x \mathrm{d}x \right| = -\int \frac{y}{1+y^2}\mathrm{d}y =$$
$$= \left| z = 1+y^2,\ \mathrm{d}z = 2y\mathrm{d}y \right| = -\frac{1}{2}\int z\mathrm{d}z = -\frac{1}{2}\log|z| + c = -\frac{1}{2}\log\left|1+\cos^2 x\right| + c$$

Příklad 5 (b) $f(x) = \dfrac{1}{\sin^3 x}$

$$\int f(x)\mathrm{d}x = \int \frac{\sin x}{\sin^4 x}\mathrm{d}x = \int \frac{\sin x}{\left(1-\cos^2 x\right)^2}\mathrm{d}x = |y = \cos x,\ \mathrm{d}y = -\sin x\mathrm{d}x| = -\int \frac{1}{(1-y^2)^2}\mathrm{d}y =$$
$$\stackrel{\text{par. zlomky}}{=} \frac{1}{4}\int \frac{1}{y-1}\mathrm{d}y - \frac{1}{4}\frac{1}{(1-y)^2}\mathrm{d}y - \frac{1}{4}\int \frac{1}{y+1}\mathrm{d}y - \frac{1}{4}\int \frac{1}{(1+y)^2}\mathrm{d}y =$$
$$= \frac{1}{4}\left(\log\left|\frac{y-1}{y+1}\right| + \frac{1}{y-1} + \frac{1}{y+1}\right) + c = \frac{1}{4}\left(\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{\cos x - 1} + \frac{1}{\cos x + 1}\right) + c$$

Příklad 5 (c) $f(x) = \dfrac{3\sin^2 x + \cos^2 x}{\sin^2 x + 3\cos^2 x}$

$$\int f(x)\mathrm{d}x = \int \frac{1+2\sin^2 x}{1+2\cos^2 x}\mathrm{d}x =$$
$$= \left| R(u,v) = \frac{1+2u^2}{1+2v^2},\ y = \operatorname{tg} x,\ \frac{1}{1+y^2}\mathrm{d}y = \mathrm{d}x,\ \sin^2 x = \frac{y^2}{1+y^2},\ \cos^2 x = \frac{1}{1+y^2} \right| =$$
$$= \int \frac{1+2\frac{y^2}{1+y^2}}{1+2\frac{1}{1+y^2}}\frac{1}{1+y^2}\mathrm{d}y = \int \frac{1+3y^2}{1+y^2}\frac{1+y^2}{3+y^2}\frac{1}{1+y^2}\mathrm{d}y = \int \frac{\left(1+3y^2\right)\left(1+y^2\right)}{3+y^2}\mathrm{d}y =$$
$$\stackrel{\text{par. zlomky}}{=} \int 3y^2 - 5 + \frac{16}{y^2+3}\mathrm{d}y = y^3 - 5y + \frac{16}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}} + c =$$
$$= \operatorname{tg}^3 x - 5\operatorname{tg} x + \frac{16}{\sqrt{3}}\arctan\frac{\operatorname{tg} x}{\sqrt{3}} + c$$

Pro $x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi), k \in \mathbb{Z}$.

Příklad 5 (d) $f(x) = \dfrac{1}{\sin x + \operatorname{tg} x}$

$$\int f(x)\mathrm{d}x = \int \frac{\cos x}{\sin x \cos x + \sin x}\mathrm{d}x = \left| R(u,v) = \frac{v}{uv+u},\ y = \cos x,\ \mathrm{d}y = -\sin x\mathrm{d}x \right| =$$
$$= \int \frac{y}{(y^2-1)(1+y)}\mathrm{d}y \stackrel{\text{par. zlomky}}{=} -\frac{1}{4}\int \frac{1}{y+1}\mathrm{d}y + \frac{1}{2}\int \frac{1}{(y+1)^2}\mathrm{d}y + \frac{1}{4}\int \frac{1}{y-1}\mathrm{d}y =$$
$$= \frac{1}{4}\log\left|\frac{y-1}{y+1}\right| - \frac{1}{2(y+1)} + c = \frac{1}{4}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| - \frac{1}{2(\cos x + 1)} + c$$

Příklad 5 (e) $f(x) = \dfrac{\cos 2x}{\sin^2 x + 2\cos^2 x}$

$$\int f(x)\mathrm{d}x = \int \frac{1-2\sin^2 x}{2-\sin^2 x}\mathrm{d}x = \left| y = \operatorname{tg} x,\ \mathrm{d}x = \frac{1}{1+y^2}\mathrm{d}y,\ \sin^2 x = \frac{y^2}{1+y^2} \right| = \int \frac{1-2\frac{y^2}{1+y^2}}{2-\frac{y^2}{1+y^2}} \frac{1}{y^2+1}\mathrm{d}y =$$
$$= \int \frac{1-y^2}{(2+y^2)(y^2+1)}\mathrm{d}y \overset{\text{par. zlomky}}{=} -3\int \frac{1}{2+y^2}\mathrm{d}y + 2\int \frac{1}{y^2+1}\mathrm{d}y =$$
$$= -\frac{3}{\sqrt{2}}\arctan\frac{y}{\sqrt{2}} + 2\arctan y + c = -\frac{3}{\sqrt{2}}\arctan\frac{\operatorname{tg} x}{\sqrt{2}} + 2\arctan \operatorname{tg} x + c =$$
$$= -\frac{3}{\sqrt{2}}\arctan\frac{\operatorname{tg} x}{\sqrt{2}} + 2x + c$$

Pro $x \in (-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), k \in \mathbb{Z}$**.**

Příklad 5 (f) $f(x) = \dfrac{1}{\sin^2 x \cdot \cos^2 x}$

$$\int f(x)\mathrm{d}x = |y = \operatorname{tg} x| = \int \frac{1}{\frac{y^2}{1+y^2}\cdot\frac{1}{1+y^2}} \cdot \frac{1}{y^2+1}\mathrm{d}y = \int \frac{1+y^2}{y^2}\mathrm{d}y = \int \frac{1}{y^2} + 1\mathrm{d}y =$$
$$= -\frac{1}{y} + y + x + c = \frac{y^2-1}{y} + c = \frac{\frac{\sin^2 x}{\cos^2 x} - 1}{\frac{\sin x}{\cos x}} + c = \frac{\sin^2 x - \cos^2 x}{\cos^2 x}\cdot\frac{\cos x}{\sin x} + c =$$
$$= \frac{-\cos(2x)}{\frac{\sin 2x}{2}} + c = \frac{-1}{2}\operatorname{cotg}(2x) + c$$

Pro $x \in (-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi), k \in \mathbb{Z}$**.**

Příklad 5 (g) $f(x) = \operatorname{tg}^5 x$

$$\int f(x)\mathrm{d}x = \int \frac{\sin^5 x}{\cos^5 x}\mathrm{d}x = |y = \cos x,\ \mathrm{d}y = -\sin y\mathrm{d}x| = -\int \frac{\left(1-y^2\right)^2}{y^5}\mathrm{d}y = -\int \frac{1}{y} - 2\frac{1}{y^3} + \frac{1}{y^5}\mathrm{d}y =$$
$$= -\log|y| - \frac{1}{y^2} + \frac{1}{6y^6} + c = -\log|\cos x| - \frac{1}{(\cos x)^2} + \frac{1}{6(\cos x)^6} + c$$

K Příkladu 6: **subtituce vybíráme podle návodu z teorie (rozbor fce R...)**

Příklad 6 (a) $f(x) = \dfrac{1+\sqrt[3]{x+1}}{\sqrt{x+1}+x+1}$

$$\begin{aligned}\int f(x)\mathrm{d}x &= \left| y=\sqrt[6]{x+1},\ \mathrm{d}y=\frac{1}{6}(x+1)^{-\frac{5}{6}} \right| = \int \frac{1+y^2}{y^3+y^6}6y^5\mathrm{d}y = 6\int\frac{y^7+y^5}{y^6+y^3}\mathrm{d}y = \\
&= 6\int \frac{y\left(y^6+y^3\right)}{y^6+y^3}+\frac{y^5-y^4}{y^6+y^3}\mathrm{d}y \overset{\text{par. zlomky}}{=} \frac{6}{2}y^2+6\int\frac{2}{3}\frac{1}{y+1}+\frac{1}{3}\frac{y-2}{y^2-y+1}\mathrm{d}y = \\
&= 3y^2+4\log|y+1|+\int\frac{2y-1}{y^2-y+1}+\frac{-3}{y^2-y+1}\mathrm{d}y = \\
&= 3y^2+4\log|y+1|+\log|y^2-y+1|-3\int\frac{1}{\left(y-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}y = \\
&= 3y^2+4\log|y+1|+\log|y^2-y+1|-3\frac{4}{3}\int\frac{1}{\left(\frac{y-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1}\mathrm{d}y = \\
&= 3y^2+4\log|y+1|+\log|y^2-y+1|-4\sqrt{\frac{3}{4}}\arctan\frac{y-\frac{1}{2}}{\sqrt{\frac{3}{4}}}+c = \\
&= 3y^2+4\log|y+1|+\log|y^2-y+1|-2\sqrt{3}\arctan\frac{2y-1}{\sqrt{3}}+c = \\
&= 3\sqrt[3]{x+1}+4\log\left|\sqrt[6]{x+1}+1\right|+\log\left|\sqrt[3]{x+1}-\sqrt[6]{x+1}+1\right|-2\sqrt{3}\arctan\frac{2\sqrt[6]{x+1}-1}{\sqrt{3}}+c\end{aligned}$$

Příklad 6 (b) $f(x)=\sqrt{\frac{x-1}{x+1}}\frac{1}{x}$

V subsitutic níže platí:

$$\begin{aligned}y^2 &= \frac{x-1}{x+1} \\
(x+1)y^2 &= x-1 \\
x\left(y^2-1\right) &= -1-y^2 \\
x &= \frac{-1-y^2}{y^2-1} \qquad \text{pro } y^2\neq 1,\ x-1\neq x+1, \quad \text{tedy pro všechna } x\neq -1\end{aligned}$$

$$\begin{aligned}\int f(x)\mathrm{d}x &= \left| y=\sqrt{\frac{x-1}{x+1}},\ \mathrm{d}y=\frac{1}{2}\sqrt{\frac{x+1}{x-1}}\frac{2}{(x+1)^2}\mathrm{d}x,\ x=\frac{-1-y^2}{y^2-1} \right| = \\
&= \int y\cdot\frac{y^2-1}{-1-y^2}y\cdot\left(\frac{-1-y^2}{y^2-1}+1\right)^2\mathrm{d}y = \int\frac{y^2\left(y^2-1\right)}{-1-y^2}\left(\frac{-1-y^2+y^2-1}{y^2-1}\right)^2\mathrm{d}y = \\
&= \int\frac{y^2\left(y^2-1\right)}{-1-y^2}\frac{4}{\left(y^2-1\right)^2}\mathrm{d}y = \int\frac{4y^2}{\left(-1-y^2\right)\left(y^2-1\right)}\mathrm{d}y = \\
&\overset{\text{par. zlomky}}{=} \int\frac{1}{y+1}+\frac{1}{y-1}-2\frac{1}{y^2+1}\mathrm{d}y = \log|y+1|+\log|y-1|-2\arctan y+c = \\
&= \log\left|\sqrt{\frac{x-1}{x+1}}-1\right|+\log\left|\sqrt{\frac{x-1}{x+1}}+1\right|-2\arctan\sqrt{\frac{x-1}{x+1}}+c\end{aligned}$$

Příklad 6 (c) $f(x)=\frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}$

$$\int f(x)\mathrm{d}x = \left| y = \sqrt[6]{x+1},\ \mathrm{d}y = \frac{1}{6}(x+1)^{-\frac{5}{6}} \right| = 6\int \frac{1-y^3}{1+y^2} y^5 \mathrm{d}y = 6\int \frac{-y^8+y^5}{1+y^2}\mathrm{d}y =$$

$$= 6\int -y^6 + y^4 + y^3 - y^2 - y + 1 + \frac{y-1}{y^2+1}\mathrm{d}y =$$

$$= 6\left(-\frac{y^7}{7} + \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} - \frac{y^2}{2} + y + \frac{1}{2}\int \frac{2y}{y^2+1} - \frac{2}{y^2+1}\mathrm{d}y\right) =$$

$$= -\frac{6y^7}{7} + \frac{6y^5}{5} + \frac{3y^4}{2} - 2y^3 - 3y^2 + 6y + 3\log(y^2+1) - 6\arctan y + c$$

Příklad 6 (d) $f(x) = \dfrac{1}{2+3\sqrt{x^2-5x+6}}, \quad x > 3$

Platí: $x^2 - 5x + 6 = (x-2)(x-3)$.

$$\int f(x)\mathrm{d}x = \left| y = \sqrt{\frac{x-2}{x-3}},\ \mathrm{d}y = \frac{1}{2}\sqrt{\frac{x-3}{x-2}}\frac{-1}{(x-3)^2}\mathrm{d}x,\ x = \frac{3y^2-2}{y^2-1} \right| =$$

$$= -2\int \frac{1}{2+3\frac{y}{y^2-1}} y \left(\frac{3y^2-2}{y^2-1} - 3\right)^2 \mathrm{d}y =$$

$$= -2\int \frac{y}{\frac{2y^2-2+3y}{y^2-1}} \frac{\left(3y^2-2-3y^2+3\right)^2}{\left(y^2-1\right)^2}\mathrm{d}y =$$

$$= -2\int \frac{y(y^2-1)}{2y^2+3y-2}\frac{1}{\left(y^2-1\right)^2}\mathrm{d}y = \int \frac{-2y}{(y^2-1)(2y^2+3y-2)}\mathrm{d}y =$$

$$\overset{\text{par. zlomky}}{=} \int \frac{1}{3(y+1)} - \frac{4}{15(y+2)} + \frac{8}{15(2y-1)} - \frac{1}{3(y-1)}\mathrm{d}y =$$

$$= \frac{1}{3}\log\left|\frac{y+1}{y-1}\right| + \frac{4}{15}\log\left|\frac{y-\frac{1}{2}}{y+2}\right| + c = \frac{1}{3}\log\left|\frac{\sqrt{\frac{x-2}{x-3}}+1}{\sqrt{\frac{x-2}{x-3}}-1}\right| + \frac{4}{15}\log\left|\frac{\sqrt{\frac{x-2}{x-3}}-\frac{1}{2}}{\sqrt{\frac{x-2}{x-3}}+2}\right| + c$$